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УРАВНЕНИЕ ЛАГРАНЖА–ЯКОБИ И ДИСПЕРСИЯ СКОРОСТЕЙ ДЛЯ РАСШИРЯЮЩИХСЯ ГРАВИТИРУЮЩИХ СИСТЕМ

The Lagrange–Jacobi Equation and Velocity Dispersion for Expanding Gravitating Systems

Резюме. Уравнение Лагранжа–Якоби исследовано для гравитирующих систем, расширяющихся по закону Хаббла. Найдено выражение для дисперсии скоростей. Рассмотрены также случаи, когда система погружена в гравитирующий или антигравитирующий фон.

Abstract. The Lagrange–Jacobi equation has been studied for gravitating systems that expand in accordance with the Hubble law. An expression for the velocity dispersion has been found. The cases are also considered when the system is embedded into a gravitating or antigravitating background.

§ 1. It is known that the Lagrange–Jacobi equation

$$\frac{1}{2}\,\ddot{I} = 2T + W \tag{1}$$

is valid for a system of N gravitating point masses. Here I is the inertia moment (relative to the barycentre), T the kinetic energy, W the potential energy. One can write that $I = MR_\mathrm{i}^2$, where M is the mass of the system, R_i its inertia radius. Then

$$\dot{I} = 2MR_\mathrm{i}\dot{R}_\mathrm{i}$$

(under natural assumption that the mass is conserved). In this paper we will consider non-steady systems which expand according to the Hubble law,

$$\dot{R}_\mathrm{i} = HR_\mathrm{i}, \qquad H = \mathrm{const}.$$

Then $\dot{I} = 2HMR_\mathrm{i}^2 = 2HI$, $\ddot{I} = 4H^2I$, and

$$2H^2I = 2T + W. \tag{2}$$

Set $T = Ms^2/2$, $W = -GM^2/R_\mathrm{g}$, where s^2 is the dispersion of total velocities averaged over the system and R_g can be called the ‘gravitational radius’. Then

$$s^2 = \frac{GM}{R_\mathrm{g}} + 2H^2R_\mathrm{i}^2. \tag{3}$$

Hence, a positive term $2H^2R_\mathrm{i}^2$ must be added to the ‘virial’ velocity dispersion GM/R_g in the RHS of Eq. (3), and we see that the velocity dispersion for non-steady (both expanding and contracting) systems is larger than for stationary ones, in accordance with ideas of Ambartsumian V. A. (La structure et évolution de l'univers, Bruxelles, 1958, p.241, § 6).

Formally, Eqs (2), (3) resemble the equation of Bouvier P. and de la Reza R. (C. r. Soc. phys. et hist. natur. Génève, 1967, p. 178). But actually these authors considered a system on an expanding background (not an expanding system).

§ 2. The velocity dispersion $s^2 = \sigma^2 + v^2$, where σ^2 is the mean dispersion of residual velocities, and v the mean expansion speed. We will consider the system as a continuous gravitating fluid, $\varrho(\mathbf{r})$ being its density. It is known that the Lagrange–Jacobi equation (1), (2) is valid for such model. Then

$$Mv^2 = \int \varrho(\mathbf{r})\mathbf{v}^2(\mathbf{r})\, d^3\mathbf{r},$$

$\mathbf{v}(\mathbf{r})$ being a local expansion speed. It is easy to find that for the simplest model of a uniform sphere $Mv^2 = H^2 I$. Recall that now $R_\mathrm{i}^2 = (3/5)R^2$, $R_\mathrm{g} = (5/3)R$, where R is the radius of the sphere. Hence $M\sigma^2 + H^2 I = (3/5)GM^2/R + 2H^2 I$. The dispersion of residual velocities $\sigma^2 = (3/5)GM/R + H^2(I/M)$.

For nonuniform systems one can set $Mv^2 = \lambda H^2 I$ with a dimensionless factor $\lambda < 1$ for realistic models. Then

$$\sigma^2 = \frac{GM}{R_\mathrm{g}} + (2-\lambda)H^2(I/M). \qquad (4)$$

Note that $I/(-W) = MR_\mathrm{i}^2/(GM^2/R_\mathrm{g})$ has the order of the crossing time (Ossipkov L.P., Astr. Rep., 2006, **50**, 116). Denote

$$\tau_\mathrm{c} = \sqrt{\frac{R_\mathrm{g}R_\mathrm{i}^2}{GM}}\,, \qquad \tau_\mathrm{e} = 1/H$$

(τ_e is the dissolution time). Rewrite Eq. (4):

$$\sigma^2 = \frac{GM}{R_\mathrm{g}}\left[1 + (2-\lambda)(\tau_\mathrm{c}/\tau_\mathrm{e})^2\right]. \qquad (5)$$

To estimate the correction to the velocity dispersion in Eq. (5) we set $R_\mathrm{g} = 5$ Mpc, $M = 2\times 10^{46}$ g (a cluster of galaxies). Then $\tau_\mathrm{c} \approx 1.6\times 10^{15}$ s. If H is of the same order as the local Hubble constant, then $\tau_\mathrm{e} \approx 10^{17}$ s. Hence $(\tau_\mathrm{c}/\tau_\mathrm{e})^2 \ll 1$ and the correction for non-stationarity has no significance for estimates of virial masses for clusters of galaxies.

§ 3. Clusters of galaxies are usually considered as embedded into dark matter halos. Duboshin G. N. and Rybakov A. I. (Sov. Astron., 1970, **13**, 704) found that for a gravitating system embedded into a uniform background whose density is ϱ_b

$$\frac{1}{2}\ddot{I} = 2T + W - \omega^2 I, \qquad \omega^2 = (4/3)\pi G\varrho_b \qquad (6)$$

(see also Duboshin G. N., Rybakov A. I., Kalinina E. N., Kholopov P. N., Soobshch. Sternberg Astron. Inst., 1971, No. 175, 31; Ossipkov L. P., Star clusters and problems of stellar evolution, Sverdlovsk, 1983, 20). Then (under assumption of the Hubble expansion)

$$\sigma^2 = \frac{GM}{R_\mathrm{g}}\{1 + [(2-\lambda)H^2 + \omega^2]\tau_\mathrm{c}^2\}.$$

It seems surprising that the presence of the background 'heats' the system.

§ **4**. Studies of many clusters of galaxies showed that outer parts of these systems are expanding (e. g. Karachentsev I. D., Nasonova O. G., MNRAS, 2010, **408**, 1075, 2010; Karachentsev I. D., Nasonova O. G., Courtois H. M., ApJ, 2011, **742**, A123). Chernin A. D. (Physics-Uspekhi, 2008, **52**, 267) (see also Chernin A. D. et al., AA, 2010, **520**, A104) suggested to explain this phenomenon by the presence of 'dark energy' that accounts for about 70 % of the total energy density in the present-day Universe. In the Newtonian approximation a cluster of galaxies should be considered as embedded into a homogeneous background of negative density (e.g., McVittie G. C. General Relativity and Cosmology, London, 1956, § 9.5). Some authors tried to generalise the virial theorem for such systems (Jackson J.C., MNRAS, 1970, **149**, 249; Novakowski M., Sanabria J. C., García A., Phys. Rev. D., 2002, **66**, 023003; Chernin A. D. et al. Gravitation & Cosmology, 2012, **18**, 1). The generalised Lagrange–Jacobi equation reads as follows:

$$\frac{1}{2}\ddot{I} = 2T + W + 2A^2 I \tag{7}$$

with $A^2 = (4/3)\pi G\varrho_\Lambda$, where $(-2\varrho_\Lambda)$ is the effective gravitating density of dark energy. Then

$$\sigma^2 = \frac{GM}{R_\mathrm{g}}\{1 + [(2-\lambda)H^2 - 2A^2]\tau_\mathrm{c}^2\}.$$

We see that a correction for non-stationarity vanishes when ϱ_Λ is equal to the critical density

$$\varrho_\mathrm{c} = \frac{3(2-\lambda)}{8\pi G}\,H^2.$$

If $\varrho_\Lambda > \varrho_\mathrm{c}$ the real velocity dispersion will be less than the virial one.

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